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ASYMPTOTIC ANALYSIS OF THE DYNAMIC PROBLEM OF THE THEORY OF ELASTICITY FOR A TRANSVERSE ISOTROPIC HOLLOW CYLINDER

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The forced vibrations of a transverse isotropic hollow cylinder, loaded by an axisymmetric harmonic force at the butt ends, are studied using the homogeneous solutions method. A dispersion equation for the corresponding three-dimensional elastic problem is derived that relates the wave number to the frequency of oscillations. This equation is solved by an asymptotic method in the case of a thin wall and the possible waveforms of the tube are found.

1. INTRODUCTION

Three-dimensional dynamic problems for solid and hollow elastic cylinders have been considered in earlier studies. Free vibrations of isotropic elastic solid cylinders were investigated within three-dimensional linear theory in references [1, 2]. In reference [1] the unknown displacements were sought in the form of series, each term of which satisfies the governing equations of the motion. Numerical results of reference [1] are in excellent agreement with experimental data in reference [3].

The natural frequencies and modes of free vibrations of solid cylinders with polygonal cross-section were studied in reference [2] by the energy method. Different boundary conditions on the butt ends were analyzed. The numerical results were presented for the solid cylinders with square and hexagonal cross-sections; different values of the relation of the cylinder length to diameter of the typical cross-section were considered. The results of reference [2] can be used as the criterion of precision of the simplified beam theories. Dynamics of the solid circular cylinder with different boundary conditions on the butt ends was considered in reference [4]. The three-dimensional solutions given in reference [4] elucidate the region of validity of Timoshenko's shear deformation beam theory. Numerical results of this paper are in excellent agreement with experimental data [3].

Numerical analysis of free vibrations of the finite-length solid and hollow isotropic elastic cylinders with free butt ends is presented in reference [5]. The results were obtained using the finite elements method within the three-dimensional linear theory of elasticity in the lowest frequency region. For a hollow cylinder the frequencies of the free vibrations were tabulated as functions of non-dimensional thickness and length. The results of reference [5] can be used for comparison with approximate analytic solutions for both thin and thick shells.

In references [6, 7] the energy method was used for analysis of the frequencies and modes of free vibrations of finite-length hollow elastic isotropic cylinders with curvilinear free surfaces and arbitrary cross-section. The qualitative peculiarities of the natural frequencies and modes of vibrations, governed by the shape and characteristic thickness of the cylinder's cross-section, were established. For instance, it was shown that for hollow cylinders with sufficiently small wall thickness, the bending modes of vibrations are the main ones in the lowest frequency region. Note that free vibrations of cylindrical elastic isotropic thin shells with arbitrary cross-section and finite length were considered in references [8–10] on the basis of the two-dimensional (applied) Kirchhoff–Love theory [11] by the asymptotic method. It was shown, in particular, that the bending modes of vibrations, the most pronounced in the lowest frequency region, have a spatially local nature and are localized on the part of the shell's surface, having relatively small stiffness.

In references [12–14] the forced axisymmetrical vibrations of isotropic elastic hollow cylinders with free curvilinear surfaces were studied on the basis of linear dynamic equations of three-dimensional theory of elasticity by the method of homogenous solutions. It was shown that this method reveals the basic features of the three-dimensional elastic problem for an isotropic shell and represents an effective tool for solving specific boundary problems. It can be used also for the accuracy estimates of the applied (two-dimensional) theories. However, the relation between three- and two-dimensional dynamic problems for an anisotropic elastic shell is studied insufficiently. Particularly, the problem of the limit transition from the three-dimensional problem to the two-dimensional one in the dynamic theory of anisotropic elasticity is of special inrterest. Existing dynamic applied theories for the anisotropic shells are based on different simplifying assumptions and need comparative analysis of their accuracy. Such an analysis can be done only within a three-dimensional approach that also permits the areas of applicability for each of these theories to be found.

2. THE DISPERSION EQUATION

Consider the axisymmetric dynamic problem of the theory of elasticity for the transverse isotropic hollow cylinder. Introduce the cylindrical co-ordinate system r, φ, z and suppose that $r_1 \leqslant r \leqslant r_2$. $0 \leqslant \varphi \leqslant 2\pi$, $-l \leqslant z \leqslant l$. The vibrations of the cylinder are described by the following equations in terms of displacements [15]:

$$b_{11}(\Delta_0 u_\rho - u_\rho/\rho^2) + \frac{\partial^2 u_\rho}{\partial \xi^2} + (1 + b_{13})\frac{\partial^2 u_\xi}{\partial \rho\, \partial \xi} + \lambda^2 u_\rho = 0,$$

$$(1 + b_{13})\frac{\partial}{\partial \xi}\left(\frac{\partial u_\rho}{\partial \rho} + \frac{u_\rho}{\rho}\right) + \Delta_0 u_\xi + b_{33}\frac{\partial^2 u_\xi}{\partial \xi^2} + \lambda^2 u_\xi = 0. \tag{1}$$

Here dimensionless variables are introduced $(\rho, \xi, u_\rho, u_\xi) = r_0^{-1}(r, z, u_r, u_z)$; $mb_{11} = 2G_0(1 - \nu_1\nu_2)$, $mb_{33} = 2G_0E_0(1 - \nu^2)$, $b_{12} = b_{11} - 2G_0$, $E_0 = E^{-1}E_1$, $G_0 = GG_1^{-1}$, $\nu_2 = E_0^{-1}\nu_1$, $mb_{13} = 2G_0\nu_1(1 + \nu)$- dimensionless parameters; $m = 1 - \nu - 2\nu_1\nu_2$, $\Delta_0 = \partial^2/\partial\rho^2 + ((1/\rho)(\partial/\partial\rho))$. It is assumed in equation (1) that the cylinder executes harmonic vibrations and the factor $\exp(\mathrm{i}\omega t)$ is omitted.

Suppose that the lateral surface of the cylinder is unloaded; that is,

$$\sigma_r = \tau_{rz} = 0, \qquad \rho = \rho_n \quad (n = 1, 2). \tag{2}$$

Arbitrary boundary conditions, changing in time with the frequency ω can be adopted at the butt ends. The solution of equations (1) and (2) is sought in the form

$$u_\rho = u(\rho)\frac{\mathrm{d}m_1}{\mathrm{d}\xi}, \quad u_\xi = w(\rho)m_1(\xi), \tag{3}$$

where the function $m_1(\xi)$ satisfies the equation

$$\frac{\mathrm{d}^2 m_1}{\mathrm{d}\xi^2} - \mu^2 m_1(\xi) = 0. \tag{4}$$

From equations (1)–(3) and accounting for equation (4) gives:

$$b_{11}\left(\Delta_0 u - \frac{u}{\rho^2}\right) + (\mu^2 + \lambda^2)u + (1 + b_{13})\frac{\mathrm{d}w}{\mathrm{d}\rho} = 0,$$

$$(1 + b_{13})\mu^2\left(\frac{\mathrm{d}u}{\mathrm{d}\rho} + \frac{u}{\rho}\right) + \Delta_0 w + a_0^2 w = 0, \qquad a_0^2 = b_{33}\mu^2 + \lambda^2, \tag{5}$$

$$\left(b_{11}\frac{\mathrm{d}u}{\mathrm{d}\rho} + \frac{b_{12}}{\rho}u + b_{13}w\right)\Bigg|_{\rho = \rho_n} = 0,$$

$$\left(\mu^2 u + \frac{\mathrm{d}w}{\mathrm{d}\rho}\right)\Bigg|_{\rho = \rho_n} = 0. \tag{6}$$

In equations (6) the relations of elasticity for transverse isotropic cylinder are used [15]. The solution of equations (5) can be represented in the form

$$u(\rho) = (a_0^2 - \alpha_1^2)Z_1(\alpha_1\rho) + (a_0^2 - \alpha_2^2)Z_1(\alpha_2\rho),$$

$$w(\rho) = -(b_{13} + 1)[\alpha_1 Z_0(\alpha_1\rho) + \alpha_2 Z_0(\alpha_2\rho)]\mu^2, \tag{7}$$

where $Z_k(\alpha\rho) = c_k \mathrm{J}_k(\alpha\rho) + c_2 \mathrm{Y}_k(\alpha\rho)$ $(k = 0, 1)$, c_1, c_2 are arbitrary constants, $\alpha_n = \sqrt{t_n}$ with t_n $(n = 1, 2)$ being the roots of equation

$$t^2 - 2q_1 t + q_2 = 0,$$

$$q_1 = b_{11}^{-1}[(b_{11}b_{33} - b_{13}^2 - 2b_{13})\mu^2 + (b_{11} + 1)\lambda^2], \tag{8}$$

$$q_2 = b_{11}^{-1}(\mu^2 + \lambda^2)a_0^2, \quad \alpha_n = \pm s_n, \quad s_n = \sqrt{q_1 - (-1)^n\sqrt{q_1^2 - q_2}}.$$

From the homogenous boundary conditions (equation (6)), the following dispersion equation is given:

$$\begin{aligned}\Delta(\mu, \lambda, \rho_1, \rho_2) \equiv{}& 8\pi^{-2} l_1 l_2 a_1 a_2 g_1 g_2 + (a_2 b_1 - a_1 b_2)\{a_1 g_2[l_1 L_{10}(\alpha_2) + l_2 L_{01}(\alpha_2)]L_{11}(\alpha_1) \\ & - a_2 g_1[l_1 L_{10}(\alpha_1) + l_2 L_{01}(\alpha_1)]L_{11}(\alpha_2)\} \\ & - (a_2 b_1 - a_1 b_2)^2(\rho_1\rho_2)^{-1}L_{11}(\alpha_1)L_{11}(\alpha_2) \\ & + a_1 a_2 g_1 g_2[L_{10}(\alpha_1)L_{01}(\alpha_2) + L_{01}(\alpha_1)L_{10}(\alpha_2)] - a_2^2 g_1^2 L_{00}(\alpha_1)L_{11}(\alpha_2) \\ & - a_1^2 g_2^2 L_{00}(\alpha_2)L_{11}(\alpha_1) = 0,\end{aligned} \tag{9}$$

$$a_n = \mu^2(a_0 + b_{13}\alpha_n^2), \quad b_n = -2G_0(a_0^2 - \alpha_n^2), \quad g_n = \alpha_n[B_0\mu^2 + b_{11}(\lambda^2 - \alpha_n^2)],$$

$$l_n(\alpha_n\rho_n)^{-1}, \quad B_0 = b_{11}b_{33} - b_{13}^2 - b_{13}, \quad L_{ij}(x) = \mathrm{J}_i(x\rho_1)\mathrm{Y}_j(x\rho_2) - \mathrm{J}_j(x\rho_2)\mathrm{Y}_i(x\rho_1),$$

$$i,j = 0,1; \quad n = 1,2.$$

3. ASYMPTOTIC ANALYSIS OF THE DISPERSION EQUATION

The left side of equation (9) as an integer function of the parameter μ has a limited set of zeros with the accumulation point at infinity. Suggesting that the cylinder is thin walled, let introduce

$$\rho_1 = 1 - \varepsilon, \quad \rho_2 = 1 + \varepsilon, \quad 2\varepsilon = \frac{r_2 - r_1}{r_0} \tag{10}$$

and assume that $\varepsilon \ll 1$. Substituting relations (10) into equation (9) gives

$$D(\mu, \lambda, \varepsilon) \equiv \Delta(\mu, \lambda, \rho_1, \rho_2) = 0. \tag{11}$$

In the case of $\lambda = O(1)$; $\varepsilon \to 0$ there are three groups of the zeros of the function $D(\mu, \lambda, \varepsilon)$: (a) two zeros $\mu_1 \sim \mu_2 \sim O(1)$; (b) four zeros, having the order $O(\varepsilon^{(-1/2)})$; (c) the finite set of the zeros having the order $O(\varepsilon^{-1})$.

The proof for this statement follows from the following schema. Expanding the function $D(\mu, \lambda, \varepsilon)$ in series with respect to the small parameter ε gives

$$D(\mu, \lambda, \varepsilon) = A\varepsilon^2[b_0 D_0(\mu, \lambda_0) + (1/3)D_1(\mu, \lambda_0)\varepsilon^2$$
$$+ (1/45)D_2(\mu, \lambda_0)\varepsilon^2 + \ldots] = 0, \tag{12}$$

where

$$A = 128(1+v)^2 m^{-1}\pi^{-2}G_0(\alpha_1^2 - \alpha_2^2)^2 a_0^2 b_0(b_{13}+1)^2, \quad b_0 = 1 - v_1 v_2, \lambda^2 = 2(1+v)\lambda_0^2,$$

$$D_0(\mu, \lambda_0) = (G_0 - \lambda_0^2)E_0 G_0 \mu^2 + \lambda_0^2(G_0 - b_0\lambda_0^2), \quad D_1(\mu, \lambda_0) = (E_0 G_0)^2\mu^6$$
$$+ 2E_0 G_0^2\{\lambda_0^2[b_0 + E_1 G_1^{-1} - v_1(1+v)] - 2(1+v)(E_0 G_0 - v_1)G_0\}\mu^4 + \{9b_0 E_0 G_0^2$$
$$+ 2G_0[2v_1(1+v) + vv_1 + 4v_1^2(1+G_0) + (v^2-3)E_0 - 2(v+3)E_0 G_0]\lambda_0^2$$
$$+ [4(1+v)b_0 E_0 G_0(2 + b_{11}^{-1}) + (b_0 - v_1 - vv_1)^2 G_0^{-1}]\lambda_0^4\}\mu^2 + 9b_0 G_0 \lambda_0^2$$
$$+ 2b_0[2(m - 2b_0)G_0 + b_0^{-1}m^2 - 4m - 2b_0]\lambda_0^4 + 2(1+v)b_0 G_0^{-1}[2G_0 + 1 - v$$
$$- 2v_1 v_2(1+G_0)]\lambda_0^6, \quad D_2(\mu, \lambda_0) = -8(1+v)b_0^{-1}(E_0 G_0 - v_1)E_0^2 G_0^2 \mu^8.$$

The cases $\lambda_0^2 = G_0$, $\lambda_0^2 = G_0 b_0^{-1}$ and $\mu = 0$ are singular ones and will be considered separately. The value of μ_k is sought from equation (12) in the form

$$\mu_k = \mu_{k0} + \varepsilon\mu_{k1} + \varepsilon^2\mu_{k2} + \cdots \quad (k = 1,2). \tag{13}$$

This gives

$$D_0(\mu_{k0}, \lambda_0) = 0, \quad \mu_{k1} = 0, \quad \mu_{k2} = [6(\lambda_0^2 - G_0)b_0 E_0 \mu_{k0}]^{-1} D_1(\mu_{k0}, \lambda_0). \tag{14}$$

Similar to the isotropic case [12], it can be shown that all other zeros of the function $D(\mu, \lambda, \varepsilon)$ tend to infinity when $\varepsilon \to 0$. They can be subdivided into two groups depending on their behaviour at $\varepsilon \to 0$: (1) $\varepsilon\mu_k \to 0$ when $\varepsilon \to 0$; (2) $\varepsilon\mu_k \to O(1)$ when $\varepsilon \to 0$. In the same way as in reference [12], it can be shown that the possibility of $\varepsilon\mu_k \to \infty$ with $\varepsilon \to 0$ is not realized.

First, find quantities μ_k corresponding to case (1). Assume that the main term of the asymptotic has the form

$$\mu_k = H_{k0}\varepsilon^{-\beta}, \quad H_{k0} = O(1), \quad 0 < \beta < 1 \tag{15}$$

and substitute equation (15) into equation (12). Retaining only the main terms gives the quantity H_{k0} in the following limiting equation:

$$[E_0 G_0 (G_0 - \lambda_0^2) H_{k0}^2 b_0 + O(\varepsilon^{2\beta})]\varepsilon^{-2\beta} + (1/3)[E_0^2 G_0^2 H_{k0}^6 + O(\varepsilon^{2\beta})]\varepsilon^{2-6\beta}$$
$$+ O[\max(\varepsilon^{4-8\beta}, \varepsilon^{2-4\beta})] = 0. \tag{16}$$

Consider three cases for the quantity β:

$$0 < \beta < 0{\cdot}5, \quad \beta = 0{\cdot}5, \quad 0{\cdot}5 < \beta < 1.$$

When $\varepsilon \to 0$, in the first and third cases from equation (16) it follows that $H_{k0} = 0$. This result contradicts the assumption in equation (15). In the second case, the equation (16) at $\varepsilon \to 0$ gives

$$E_0 G_0 H_{k0}^2 [3(G_0 - \lambda_0^2) b_0 + E_0 G_0 H_{k0}^4] = 0. \tag{17}$$

Seeking in this case the quantity μ_k in the form

$$\mu_k = \varepsilon^{-0\cdot5}(\mu_{k0} + \varepsilon^{0\cdot5}\mu_{k1} + \varepsilon\mu_{k2} + \dots) \quad (k = 3, 4, 5, 6), \tag{18}$$

equation (12) gives

$$\mu_{k0} = H_{k0}, \quad \mu_{k1} = 0, \quad \mu_{k2} = [20E_0 G_0 (G_0 - \lambda_0^2)\mu_{k0}]^{-1}\{[5b_0(2G_0 - 1) + 10E_1 G_1^{-1}$$
$$+ 2(4 - 5G_0)v_1(1 + v) - 8(1 + v)E_0 G_0]\lambda_0^4 + G_0[5 - 4(1 + v)E_0 G_0^2 + 14v_1(1 + v)G_0$$
$$- 10b_0 G_0 - 10E_1 G_0 G_1^{-1}]\lambda_0^2 + 4(1 + v)(E_0 G_0 - v_1)(5G_0 - 2)G_0^2\}. \tag{19}$$

As follows from equations (17) and (19), in the case $\lambda_0^2 < G_0$ four complex roots of equation (12) exist whilst in the case $\lambda_0^2 > G_0$ two real and two purely imaginary roots exist. The latter two roots govern solutions, propagating in the direction of the z-axis. To find the asymptotic for the third group (c), when a countable set of zeros exists, the parameter μ_k is represented in the form

$$\mu_k = \varepsilon^{-1}\delta_k + O(1) \quad (k = 7, 8, \dots). \tag{20}$$

However, as noted in references [16, 17], the parameters q_1, q_2 in equation (8) take different values depending on the characteristics of material v, v_1, v_2, G_0 and the frequency parameter λ. This leads to different asymptotic expressions for Bessel's functions in equation (7). The case under consideration gives

$$q_1 \equiv q_1^* \delta_k^2, \quad q_1^* = b_{11}^{-1}(b_{11}b_{33} - b_{13}^2 - 2b_{13}), \quad q_2 \equiv q_2^* \delta_k^4, \quad q_2^* = b_{11}^{-1} b_{33}$$

Consider the following possible cases for equation (8):

$$(\alpha)\ q_1^* > 0, \quad q_1^{*2} - q_2^* \neq 0, \quad \alpha_{1,2} = \pm s_1\delta_k, \quad \alpha_{3,4} = \pm s_2\delta_k,$$

$$s_{1,2} = \sqrt{q_1^* \pm \sqrt{q_1^{*2} - q_2^*}} \quad (q_1^{*2} > q_2^*);$$

$$s_{1,2} = \kappa \pm \mathrm{i}\eta = \sqrt{q_1^* \pm \mathrm{i}\sqrt{q_2^* - q_1^{*2}}} \quad (q_1^{*2} < q_2^*);$$

$$(\beta)\ \alpha_{1,2} = \alpha_{3,4} = p\delta_k, \quad q_1^* > 0, \quad q_1^{*2} - q_2^* = 0, \quad p = \sqrt{q_1^*},$$

$$(\gamma)\ q_1^* < 0, \quad q_1^{*2} - q_2^* \neq 0, \quad \alpha_{1,2} = \pm \mathrm{i}s_1\delta_k, \quad \alpha_{3,4} = \pm \mathrm{i}s_2\delta_k$$

$$s_{1,2} = \sqrt{|q_1^*| \pm \sqrt{q_1^{*2} - q_2^*}} \quad (q_1^{*2} > q_2^*), \quad s_{1,2} = \sqrt{|q_1^*| \pm \mathrm{i}\sqrt{q_2^* - q_1^{*2}}} \quad (q_1^{*2} < q_2^*),$$

$$(\delta)\, q_1^* < 0, \quad q_1^{*2} - q_2^* = 0, \quad \alpha_{1,2} = \alpha_{3,4} = \mathrm{i}p\delta_k, \quad p = \sqrt{|q_1^*|}.$$

For cases (α) and (β) after substitution of equation (20) into equation (9) and transformation of the result with the use of asymptotic expansions of the functions $\mathrm{J}_k(x)$, $\mathrm{Y}_k(x)$, gives, respectively,

$$(s_2 - s_1)\sin[(s_1 + s_2)\delta_k] \pm (s_2 + s_1)\sin[(s_2 - s_1)\delta_k] = 0, \tag{21}$$

$$\kappa\sin[2\eta\delta_k] \pm \eta\sinh[2\kappa\delta_k] = 0, \quad q_1^{*2} < q_2^*, \tag{22}$$

$$\sin[2p\delta_k] \pm 2p\delta_k = 0, \quad q_1^* > 0, \quad q_1^{*2} - q_2^* = 0. \tag{23}$$

For cases (γ) and (δ) similar formulae follow from the corresponding results of cases (α) and (β) by the formal replacement of s_1, s_2, p by $\mathrm{i}s_1, \mathrm{i}s_2, \mathrm{i}p$ respectively. Equations (21)–(23) coincide with equations for characteristic factors of the Saint–Venant boundary effects in the theory of transverse isotropic thick plates [16, 17]. In the same references the roots of these equations were also investigated. Properties of these roots are very important for a full description of the stress–strain state of the shell. As noted in reference [18], in the case of essential anisotropy, corresponding to large values of G_0, the Saint–Venant boundary layer damps weakly and the boundary layer solution represents, in fact, the penetrating solution. As a result, the stress–strain state of the transverse isotropic and isotropic shells is essentially different.

Note that if in equation (12) $\lambda_0 \sim \varepsilon^q (q > 0)$ is assumed, then two roots $\mu_k (k = 1, 2)$ with asymptotic behaviour $\mu_k \sim \varepsilon^q$ are given. Indeed, let

$$\mu_k = \mu_{k0}\varepsilon^\beta, \quad \lambda_0 = \Lambda\varepsilon^q, \quad \beta > 0, q > 0. \tag{24}$$

Substitution of equation (24) into equation (12) makes it clear that the constructed asymptotic process is consistent only in the case $q = \beta$. The value of μ_k is sought in the form

$$\mu_k = \mu_{k0}\varepsilon^q + \mu_{k1}\varepsilon^{3q} + \cdots, \quad \lambda_0 = \Lambda\varepsilon^q. \tag{25}$$

Equations (25) and (12) give

$$\mu_{k0} = \pm \mathrm{i}\Lambda(E_0G_0)^{-0\cdot5}, \quad \mu_{k1} = \pm 0\cdot5\mathrm{i}\nu_1^2\Lambda^3(E_0G_0)^{-1\cdot5}.$$

These roots correspond to the so-called super-low-frequency vibrations of the cylinder.

The possibility of the appearance of such vibrations in thin elastic shells has been discussed in reference [19].

Consider now the particular cases when (1) $\lambda_0^2 = G_0$ or (2) $b_0^{-1} G_0$ (the case when $\mu = 0$, corresponding to the so-called thickness resonance of the hollow cylinder [14], needs special consideration). In the case of (1) $\lambda_0^2 = G_0$ in equation (12) and μ_p is found in the form

$$\mu_p = \mu_{p0}\varepsilon^{-1/3} + \mu_{p1}\varepsilon^{1/3} + \cdots \quad (p = 1\text{–}6). \tag{26}$$

Equations (26), and (12), accounting for the relation $\lambda_0^2 = G_0$, yield

$$\mu_{p0}^6 + 3v_1 v_2 b_0 E_0^{-2} = 0, \quad \mu_{p1} = -(3E_0\mu_{p0})^{-1} G_0[b_0 + v_1(1+v)$$

$$+ E_1 G_1^{-1} - 2(1+v)E_0 G_0].$$

Note that roots of the dispersion equation, following from equations (21)–(23), remain valid in this case also. Thus, in the case $\lambda_0^2 = G_0$ six zeros of the function $D(\mu, \lambda, \varepsilon)$ are given (two of them are purely imaginary), growing as $\varepsilon^{-1/3}$ at $\varepsilon \to 0$, and the finite set of zeros, defined by equations (21)–(23).

The relation between roots, defined by equation (26), and ones, is sought following from equations (13) and (18). For such an aim it is necessary to study the behaviour of the zeros of the function $D(\mu, \lambda, \varepsilon)$ in the neighbourhood of the value $\lambda_0^2 = G_0$, assuming $\lambda_0^2 - G_0 = c_0\varepsilon^{\alpha}(\alpha > 0)$, $\mu_k = \mu_{k0}\varepsilon^{-\beta}$. Keeping only the main terms in the expansion,

$$D(\mu, \lambda, \varepsilon) = A\langle -E_0 G_0 b_0 c_0 \mu_{k0}^2 \varepsilon^{\alpha - 2\beta} + b_0(1-b_0)G_0^2 + b_0(1-2b_0)G_0 c_0 \varepsilon^{\alpha} + O(\varepsilon^{2\alpha})$$

$$+ (1/3)\{E_0^2 G_0^2 \mu_{k0}^6 \varepsilon^{2-6\beta} + 2E_0 G_0^2[b_0 + v_1(1+v)$$

$$+ E_1 G_1^{-1} - 2(1+v)E_0 G_0]\mu_{k0}^4 \varepsilon^{2-4\beta}$$

$$+ O[\max(\varepsilon^{2-2\beta}, \varepsilon^{4-8\beta})]\}\rangle = 0. \tag{27}$$

As shown in analysis of equation (27), the following cases are possible:

(1) $\alpha = 2\beta$, $0 < \alpha < 2/3$; (2) $\alpha = 2\beta$, $\alpha = 2/3$;
(3) $\alpha = 2 - 4\beta$, $1/2 < \alpha < 2/3$; (4) $\beta = 1/3$, $\alpha > 2/3$.

For case 1 the value of μ_k is sought in the form

$$\mu_k = \mu_{k0}\varepsilon^{-\alpha/2} + \mu_{k1}\varepsilon^{\alpha/2} + \cdots \quad (0 < \alpha < 1/2),$$

$$\mu_k = \mu_{k0}\varepsilon^{-\alpha/2} + \mu_{k1}\varepsilon^{2-7\alpha/2} + \cdots \quad (1/2 < \alpha < 2/3). \tag{28}$$

Substituting equation (28) into equation (27) gives

$$\mu_{k0}^2 = v_1 v_2 G_0 (c_0 E_0)^{-1}, \quad \mu_{k1} = (1 - 2b_0)(2E_0\mu_{k0})^{-1} \quad (0 < \alpha < 1/2),$$

$$\mu_{k1} = (1 - 2b_0)(2E_0\mu_{k0})^{-1} + v_1^3 v_2^3 G_0^4 (6b_0 \mu_{k0} E_0^2 c_0^2)^{-1} \quad (\alpha = 1/2),$$

$$\mu_{k1} = v_1^3 v_2^3 G_0^4 (6b_0 \mu_{k0} E_0^2 c_0^4)^{-1} \quad (1/2 < \alpha < 2/3).$$

It is seen that these roots, corresponding to similar ones, defined by equations (13) and (14), grow when $\lambda_0^2 \to G_0$. Depending on the sign of the number c_0 they can be real or purely imaginary. In case 2, from equations (27)

$$\mu_k = \mu_{k0}\varepsilon^{-1/3} + \mu_{k1}\varepsilon^{1/3} + \cdots, \tag{29}$$

where

$$\mu_{k0}^6 - 3b_0(E_0G_0)^{-1}c_0\mu_{k0}^2 + 3v_1v_2b_0E_0^{-2} = 0, \quad \mu_{k1} = [2\mu_{k0}(c_0b_0 - E_0G_0\mu_{k0}^4)]^{-1}$$
$$\{c_0b_0(1-2b_0)E_0^{-1} + (2/3)G_0^2[b_0 + v_1(1+v) + E_1G_1^{-1} - 2(1+v)E_0G_0]\mu_{k0}^4\}.$$

These relations give six roots, of which two correspond to the roots defined by equations (13) and (14) while the four remaining ones correspond to the roots defined by equation (18). These groups of zeros of the dispersion equation at $c_0 \to 0$ coincide completely with the zeros defined by equation (26).

For case 3

$$\mu_k = \mu_{k0}\varepsilon^{\alpha/4-1/2} + \mu_{k1}\varepsilon^{1/2-5\alpha/4} + \cdots, \quad \mu_{k0}^4 - 3c_0b_0(E_0G_0)^{-1} = 0,$$
$$\mu_{k1} = -v_1v_2G_0(4c_0E_0\mu_{k0})^{-1}. \quad (30)$$

Substitution of $\lambda_0^2 - G_0 = c_0\varepsilon^\alpha$ into equation (17) leads to the form of equations (30) that coincides with the one defined by equations (18) and (19). Finally, in case 4

$$\mu_k = \mu_{k0}\varepsilon^{-1/3} + \mu_{k1}\varepsilon^{\alpha-1} + \ldots, \quad \mu_{k0}^6 + 3v_1v_2b_0E_0^{-2} = 0, \quad \mu_{k1} = c_0b_0(2E_0G_0\mu_{k0})^{-1}. \quad (31)$$

At $c_0 \to 0$ these zeros coincide with the ones defined by equation (26). Note that such unusual behaviour of the roots of the dispersion equation in the isotropic case ($\lambda_0^2 \to 1$) has been investigated in reference [14].

Analysis of the modes corresponding to the roots of the dispersion equation obtained above shows that at $\lambda_0^2 < G_0$ the character of the integrals of the dynamic theory of elasticity does not differ qualitatively from the static integrals of the theory of elasticity. On the contrary, in the case $\lambda_0^2 > G_0$ there is an essential difference. Therefore, it is natural to consider the value $\lambda_0^2 = G_0$ as “the turning point”, corresponding to the change in the character of the dynamic integrals of the theory of elasticity.

For the special case (2) $\lambda_0^2 = b_0^{-1}G_0$, the zeros, defined by equation (13), vanish and the ones defined by equations (18) and (19), take the form

$$\mu_{k0}^4 - 3v_2^2 = 0, \quad \mu_{k1} = 0, \quad \mu_{k2} = (10b_0\mu_{k0})^{-1}\{4v_2^2(1+v)(E_0G_0 - v_1)$$
$$+ 10(1+v)b_0(E_0G_0 - v_1)G_0E^{-1} - 5[b_0 + E_1G_1^{-1} - v_1(1+v)]G_0E_0^{-1}\}.$$

Thus, in this case four growing zeros are obtained, two of them are purely imaginary. Note that the zeros, described by equations (21)–(23), also remain valid in this case.

Consider the case when the frequency parameter λ^2 tends to infinity when $\varepsilon \to 0$. The vibrations, corresponding to this case, following reference [19], can be called the extra-high-frequency ones. It is possible to show that all zeros of the function $D(\mu, \lambda, \varepsilon)$ tend to infinity when $\lambda \to \infty$ at $\varepsilon \to 0$.

Consider separately the following three limiting cases at $\varepsilon \to 0$: (1) $\lambda\varepsilon \to 0$; (2) $\lambda\varepsilon \to O(1)$; (3) $\lambda\varepsilon \to \infty$.

For case (1) assume that the main terms of the asymptotic for μ_k and λ_0 have the form

$$\mu_k = \mu_{k0}\varepsilon^{-\beta}, \quad \lambda_0 = \Lambda\varepsilon^{-q}, \quad \mu_{k0} = O(1), \quad \Lambda = O(1), \quad 0 < \beta < 1, \quad 0 < q < 1. \quad (32)$$

It is simple to show that the asymptotic process is consistent in this case only if the inequality $q \leqslant \beta$ is true. Consider separately the cases: $q = \beta$ and $q < \beta$. For the case $q = \beta$ the μ_k value is sought in the form

$$\mu_k = \mu_{k0}\varepsilon^{-\beta} + \mu_{k1}\varepsilon^{\beta} + \cdots \quad (0 < \beta < 1/2),$$
$$\mu_k = \mu_{k0}\varepsilon^{-\beta} + \mu_{k1}\varepsilon^{2-3\beta} + \cdots \quad (1/2 \leqslant \beta < 1), \quad \lambda_0 = \Lambda\varepsilon^{-\beta}. \quad (33)$$

Substituting these expansions into equation (12) gives

$$\mu_{k0} = \pm \mathrm{i}\Lambda(b_0 E_0^{-1} G_0^{-1})^{1/2}, \quad \mu_{k1} = v_2^2(2\mu_{k0})^{-1} \quad (0 < \beta < 1/2), \quad \mu_{k1} = v_2^2(2\mu_{k0})^{-1}$$

$$+ (6\mu_{k0} E_0 G_0)^{-1}\Lambda^4\gamma_1 \quad (\beta = 1/2), \quad \mu_{k1} = (6\mu_{k0} E_0 G_0)^{-1}\Lambda^4\gamma_1 \quad (1/2 < \beta < 1),$$

$$\gamma_1 = E_0^{-1} b_0^2(2 - G_0^{-1}) + 2E_1 G_1^{-1} b_0 - 2v_2(1 + v)b_0 - 4(1 + v)b_0(2 + b_{11}^{-1})$$

$$- (1 - v_1 - vv_1 - v_1 v_2)^2 E_0^{-1} G_0^{-2} + 2(1 - v^2)G_0^{-1} + 4(1 + v)$$

$$- 4v_1 v_2(1 + v)(1 + G_0^{-1}).$$

When $q < \beta$ equations (32) and (12) give for μ_{k0} and Λ, keeping only the main terms of expansions:

$$D(\mu, \lambda, \varepsilon) \equiv A\langle[- E_0 G_0 \Lambda^2 \mu_{k0}^2 + O(\varepsilon^{2\beta - 2q})]\, \varepsilon^{-2\beta - 2q}$$

$$+ (1/3)\{E_0^2 G_0^2 \mu_{k0}^6 + O[\max(\varepsilon^{2-2\beta}, \varepsilon^{2\beta - 2q})]\}\, \varepsilon^{2 - 6\beta}\rangle = 0.$$

It follows that $q = 2\beta - 1$. If $q > 0$, then $\beta > 1/2$. Note that the case $q = 0$, corresponding to the value $\beta = 1/2$, was investigated above.

The values of μ_k and λ_0 are sought in the form

$$\mu_k = \mu_{k0}\varepsilon^{-\beta} + \mu_{k1}\varepsilon^{3\beta - 2} + \cdots \quad (1/2 < \beta < 2/3),$$

$$\mu_k = \mu_{k0}\varepsilon^{-\beta} + \mu_{k1}\varepsilon^{2 - 3\beta} + \cdots \quad (2/3 \leqslant \beta < 1), \quad \lambda_0 = \Lambda\varepsilon^{1 - 2\beta}. \qquad (34)$$

Substituting equation (34) into equation (12) gives

$$\mu_{k0}^4 - 3b_0\Lambda^2(E_0 G_0)^{-1} = 0, \quad \mu_{k1} = -(4\Lambda^2 G_0)^{-1}\mu_{k0} \quad (1/2 < \beta < 2/3),$$

$$\mu_{k1} = -(4\Lambda^2 G_0)^{-1}\mu_{k0} - (20E_0 G_0 \mu_{k0})^{-1}\Lambda^2\gamma_2 \quad (\beta = 2/3),$$

$$\mu_{k1} = -(20E_0 G_0 \mu_{k0})^{-1}\Lambda^2\gamma_2\, (2/3 < \beta < 1), \quad \gamma_2 = 5b_0(2G_0 - 1) + 10E_1 G_0 G_1^{-1}$$

$$- 10v_1(1 + v)G_0 - 8(1 + v)(E_0 G_0 - v_1).$$

Thus, in this case there are four zeros, growing as $\varepsilon^{-\beta}$, two of which are real and two purely imaginary. Note that purely imaginary zeros correspond to the so-called irregular degeneration [20].

Note that an additional case is possible also, when

$$\mu_k = \varepsilon^{-1}\delta_k + O(\varepsilon^{1 - 2\beta}), \quad \lambda = \Lambda\varepsilon^{-\beta}, \quad 0 < \beta < 1.$$

In this case, the first term of the asymptotics receives zeros determined by relations (20)–(23).

So, in the cases $\lambda = \Lambda\varepsilon^{-\beta}$ and $\Lambda\varepsilon^{1 - 2\beta}$ correspondingly two and four zeros are obtained, growing as $\varepsilon^{-\beta}$, and the countable set of zeros, growing as ε^{-1}.

For case (2) the solution is sought in the form

$$\mu_k = \varepsilon^{-1}\delta_k + O(\varepsilon), \quad \lambda = s\varepsilon^{-1}, \qquad (35)$$

Equation (35) is substituted into equation (12) and the result is transformed to account for the asymptotic expansions of Bessel's functions for large values of the argument. It leads to the following equation for δ_k:

$$[h_{1k}\kappa_{21}\sin(h_{1k})\cos(h_{2k}) - h_{2k}\kappa_{12}\sin(h_{2k})\cos(h_{1k})]\,[h_{1k}\kappa_{21}\cos(h_{1k})\sin(h_{2k}) - h_{2k}\kappa_{12}\sin(h_{1k})\cos(h_{2k})] = 0. \tag{36}$$

Here $h_{nk} = \sqrt{\tau_{nk}}$, where τ_{nk} are roots of the quadratic equation

$$b_{11}\tau^2 - [(b_{11}b_{33} - b_{13}^2 - 2b_{13})\delta_k^2 + (1 + b_{11})s^2]\tau + (\delta_k^2 + s^2)(b_{33}\delta_k^2 + s^2) = 0 \tag{37}$$

and $\kappa_{mj}(m,j = 1,2; m \neq j)$ are determined according to the formula

$$\kappa_{mj} = (b_{11}b_{33} - b_{13}^2)(h_{mk}^2 + s^2)\delta_k^2 + b_{11}s^4 - (b_{11}h_{jk}^2 + 2b_{13}\delta_k^2 + b_{33}\delta_k^4)s^2.$$

In the derivation of equation (36) it is assumed that the roots of equation (37) are real and different. Note, that at a given value of λ equation (36) defines a countable set of roots δ_k. For the isotropic case this equation turns into the Rayleigh–Lamb equation [13].

In case (3), denoting $\varepsilon\mu_k$ by x_k and $\varepsilon\lambda$ by y, for the main term of the asymptotic equation (36) is again obtained; i.e., this equation remains true also in the case $\lambda \sim \varepsilon^{-\beta}$, $\beta > 1$.

4. ASYMPTOTIC DERIVATION OF THE HOMOGENEOUS SOLUTIONS

Assuming that ε is the small parameter of the problem, the homogeneous solutions will be sought, corresponding to different groups of the roots of the dispersion equation. The main terms of the solutions, corresponding to the roots in equations (13)–(19) and in equations (25)–(34), coincide with the well-known solutions of applied (two-dimensional) theory of shells and are not presented here.

Consider first the case corresponding to the roots in equations (21)–(23). Using the relation $\rho = 1 + \varepsilon\eta^*$, $-1 \leqslant \eta^* \leqslant 1$ and expanding solutions into series with respect to the small parameter ε, roots are described by equation (21) (from here only amplitudes of the displacements are presented; the stresses can be calculated according to the generalized Hooke's law):

$$u_{\rho k} = \varepsilon b_{11}^{-1}b_{33}[(B_0 - b_{11}s_1^2)\cos(s_2\delta_k)\cos(s_1\delta_k\eta^*) - (B_0 - b_{11}s_2^2)\cos(s_1\delta_k)\cos(s_2\delta_k\eta^*) + O(\varepsilon)]\frac{\mathrm{d}m_k}{\mathrm{d}\xi} \quad (k = 1,3,5,\ldots), \tag{38}$$

$$u_{\xi k} = \delta_k[s_1(b_{33} + b_{13}s_2^2)\cos(s_2\delta_k)\sin(s_1\delta_k\eta^*) - s_2(b_{33} + b_{13}s_1^2)\cos(s_1\delta_k)\sin(s_2\delta_k\eta^*) + O(\varepsilon)]m_k,$$

where δ_k are roots of the equation

$$(s_2 - s_1)\sin[(s_2 + s_1)\delta_k] - (s_2 + s_1)\sin[(s_2 - s_1)\delta_k] = 0.$$

Similarly, for the case in equatiion (22)

$$u_{pk} = \varepsilon[(b_{33} + \kappa^2 - \eta^2)\Delta_1 \cosh(\kappa\delta_k\eta^*)\cos(\eta\delta_k\eta^*) + 2\kappa\eta\sinh(\kappa\delta_k\eta^*)\sin(\eta\delta_k\eta^*)\Delta_2 + O(\varepsilon)]\frac{\mathrm{d}m_k}{\mathrm{d}\xi} \quad (k = 1, 3, 5, \dots) \tag{39}$$

$$u_{\xi k} = -(b_{31} + 1)\delta_k[\alpha\Delta_1 \sinh(\kappa\delta_k\eta^*)\cos(\eta\delta_k\eta^*) + \eta\Delta_2\cosh(\kappa\delta_k\eta^*)\sin(\eta\delta_k\eta^*) + O(\varepsilon)]m_k,$$

where

$$\Delta_1 = -\kappa\eta(b_{13} - 1)\sinh(\kappa\delta_k)\sin(\eta\delta_k) + \eta^2(b_{13} + 1)\cosh(\kappa\delta_k)\cos(\eta\delta_k),$$

$$\Delta_2 = (b_{33} - b_{13}\kappa^2 - \eta^2)\cosh(\kappa\delta_k)\cos(\eta\delta_k) + \kappa\eta(b_{13} + 1)\sinh(\kappa\delta_k)\sin(\eta\delta_k).$$

Finally, for the case of equation (23) it follows that

$$u_{pk} = \varepsilon p[(p\delta_k\sin(p\delta_k) - \frac{b_{13} + 2}{b_{13} + 1}\cos(p\delta_k))\cos(p\delta_k\eta^*) - \eta^* p\delta_k\cos(p\delta_k)\sin(p\delta_k\eta^*) + O(\varepsilon)]\frac{\mathrm{d}m_k}{\mathrm{d}\xi} \quad (k = 1, 3, 5, \dots), \tag{40}$$

$$u_{\xi k} = \delta_k[(p\delta_k\sin(p\delta_k) - \frac{1}{b_{13} + 1}\cos(p\delta_k))\sin(p\delta_k\eta^*) + \eta^* p\delta_k\cos(p\delta_k)\cos(p\delta_k\eta^*) + O(\varepsilon)]m_k(\xi).$$

The expressions corresponding to the even values of k can be obtained from equations (38)–(40) by replacing $\cos x$ by $\sin x$, $\sin x$ by $-\cos x$, $\cosh x$ by $\sinh x$ and $\sinh x$ by $-\cosh x$ respectively.

When $\lambda\varepsilon \to O(1)$ with $\varepsilon \to 0$, the displacements in a thin hollow cylinder, executing extra-high-frequency vibrations, are described by the expressions

$$u_{pk} = \varepsilon[(b_{33}\delta_k^2 + s^2 - h_{1k}^2)(b_{33}\delta_k^2 + b_{13}h_{2k}^2 + s^2)\cos(h_{2k})\cos(h_{1k}\eta^*) - (b_{33}\delta_k^2 + s^2 - h_{2k}^2)(b_{33}\delta_k^2 + b_{13}h_{1k}^2 + s^2)\cos(h_{1k})\cos(h_{2k}\eta^*) + O(\varepsilon)]\frac{\mathrm{d}m_k}{\mathrm{d}\xi},$$

$$u_{\xi k} = (b_{13} + 1)\delta_k^2[(b_{33}\delta_k^2 + b_{13}h_{2k}^2 + s^2)\cos(h_{2k})\sin(h_{1k}\eta^*) - (b_{33}\delta_k^2 + b_{13}h_{1k}^2 + s^2)\cos(h_{1k})\sin(h_{2k}\eta^*) + O(\varepsilon)]m_k(\xi) \quad (k = 1, 3, 5, \dots), \tag{41}$$

where δ_k are the roots of the equation

$$h_{1k}\kappa_{21}\sin(h_{1k})\cos(h_{2k}) - h_{2k}\sin(h_{2k})\cos(h_{1k}) = 0.$$

This equation follows from equation (37) taking into account the relation $h_{nk} = \sqrt{\tau_{nk}}$. Formulae, corresponding to the even values of k follow from equation (41) by replacing $\cos x$ for $\sin x$ and $\sin x$ for $-\cos x$ respectively.

It is important to note that solutions, defined by equations (38)–(41), cannot be determined from the applied theory of shells. The role of these solutions in the isotropic theory of shells has been discussed in detail in reference [14].

Similar to the isotropic case [14], it can be proved that for the transversely isotropic cylinder the generalized orthogonality condition is true also:

$$\int_{\rho_1}^{\rho_2} (u_{rp}\tau_{rz}^k - \sigma_{zp}u_{zk})\rho\,\mathrm{d}\rho = 0 \quad (p \neq k). \tag{42}$$

Note that this condition does not allow the boundary conditions at the butt ends to be satisfied exactly. Apparently, in the general case only the method of reducing to the infinite systems of algebraic linear equations can be suggested. Nevertheless, for some special cases at the butt ends the generalized orthogonality condition permits to present the solution in the form of a series whose coefficients can be defined exactly [14].

Furthermore, the condition in equation (42) can be useful for solving the infinite system of equations because it always permits at least one of the boundary conditions at the butt end to be satisfied exactly.

The process of reducing the boundary problem of the theory of elasticity to solving the infinite algebraic system is well known and is not discussed here. In the present article, the homogenous solutions are obtained that can satisfy the boundary conditions at the butt ends of a cylinder. The case of non-homogenous boundary conditions at the lateral surface of the cylinder can be studied by the methods developed in references [21, 22].

5. COMPARSION WITH SOME APPLIED THEORIES

The results of analysis of the dispersion equation (12) are compared here with the ones obtained by the Kirchhoff–Love and Ambartsumjan applied theories [23]. Note that the transverse-isotropic shell described in reference [23] in the co-ordinate system z, φ, r, corresponds to the orthotropic one in the co-ordinate system r, φ, z, used in the present article.

The equations of motion in displacements for the axisymmetric case in the Kirchhoff–Love theory have the form [23]

$$c_{11}\frac{\partial^2 u_0}{\partial \xi} + c_{12}\frac{\partial w_0}{\partial \xi} = \frac{hgr_0^2}{G_1}\frac{\partial^2 u_0}{\partial t^2}; \quad c_{12}\frac{\partial u_0}{\partial \xi} + \left(w + \frac{D_{11}}{r_0^2}\frac{\partial^4 w_0}{\partial \xi^4}\right) = -\frac{hgr_0^2}{G_1}\frac{\partial^2 w_0}{\partial t^2}. \tag{43}$$

Here $u_0 = u_0(\xi, t)$, $w_0 = w_0(\xi, t)$ are displacements of the shell middle surface in the longitudinal and transverse directions; correspondingly $c_{11} = 2(1+v)G_0E_0hb_0^{-1}$, $c_{22} = 2(1+v)G_0hb_0^{-1}$, $c_{12} = 2v_1(1+v)G_0hb_0^{-1}$, $D_{11} = c_{11}h^2/3$, where h is the thickness of the shell. The solution of equation (43) is in the form

$$u_0 = A_1 \exp(\mu\xi + \mathrm{i}\omega t), \quad w_0 = B_1 \exp(\mu\xi + \mathrm{i}\omega t).$$

It leads to the following dispersion equation:

$$D^{(Kr)}(\mu, \lambda_0, \varepsilon) \equiv b_0 D_0^{(Kr)}(\mu, \lambda_0) + (1/3)D_1^{(Kr)}(\mu, \lambda_0)\varepsilon^2 = 0, \tag{44}$$

where

$$D_0^{(Kr)}(\mu, \lambda_0) = D_0(\mu, \lambda_0), \quad D_1^{(Kr)}(\mu, \lambda_0) = E_0^2G_0^2\mu^6 + b_0E_0G_0\lambda_0^2\mu^4 \neq D_1(\mu, \lambda_0).$$

For the sake of comparison the results of asymptotic analysis of the dispersion equation (44) for some characteristic cases are presented. From equation (44) the following groups of zeros for the function $D^{(Kr)}(\mu, \lambda_0, \varepsilon)$ can be obtained:

$$(1)\ \mu_k = \mu_{k0} + \mu_{k2}\varepsilon^2 + \ \dots \quad (k = 1, 2), \quad D_0^{(Kr)}(\mu_{k0}, \lambda_0) = D_0(\mu_{k0}, \lambda_0) = 0, \tag{45}$$

$$\mu_{k2} = [6b_0E_0G_0(\lambda_0^2 - G_0)\mu_{k0}]^{-1}D_1^{(Kr)}(\mu_{k0}, \lambda_0) \neq [6b_0E_0G_0(\lambda_0^2 - G_0)\mu_{k0}]^{-1}D_1(\mu_{k0}, \lambda_0),$$

$$(2)\ \mu_k = \mu_{k0}\varepsilon^q + \mu_{k1}\varepsilon^{3q} + \ \cdots, \quad \lambda_0 = \Lambda\varepsilon^q, \quad q > 0, \tag{46}$$

$$\mu_{k0} = \ \pm \mathrm{i}\Lambda(E_0G_0)^{-1/2}, \quad \mu_{k1} = \ \pm \mathrm{i}v_1^2\Lambda^3(1/2)(E_0G_0)^{-3/2},$$

$$(3)\ \mu_k = \varepsilon^{-1/2}(\mu_{k0} + \varepsilon\mu_{k2} + \ \cdots), \quad \mu_{k0}^4 + 3b_0(G_0 - \lambda_0^2)(E_0G_0)^{-1} = 0, \tag{47}$$

$$\mu_{k2} = v_2^2\lambda_0^2[4(G_0 - \lambda_0^2)\mu_{k0}]^{-1}.$$

In the case of the Ambartsumjan theory the equations of motion in terms of displacements for the axisymmetric case have the form [23]

$$c_{11}\frac{\partial u_0}{\partial \xi^2} + c_{12}\frac{\partial w_0}{\partial \xi} = \frac{ghr_0^2}{G_1}\frac{\partial^2 u_0}{\partial t^2}, \quad c_{12}\frac{\partial u_0}{\partial \xi} + c_{22}w - \frac{h^3r_0}{12}\frac{\partial \varphi}{\partial \xi} = \ -\frac{ghr_0^2}{G_1}\frac{\partial^2 w_0}{\partial t^2},$$

$$D_{11}\frac{\partial^2 w_0}{\partial \xi^3} - \frac{h^3r_0}{12}D_{11}\frac{\partial^2 \varphi}{\partial \xi^2} + \frac{h^3r_0^3}{12}\varphi = 0, \tag{48}$$

where $\varphi = \varphi(\xi, t)$ is the shear deformation function. The solution of equations (48) is sought in the form

$$u_0 = A_2\exp(\mu\xi + \mathrm{i}\omega t), \quad w_0 = B_2\exp(\mu\xi + \mathrm{i}\omega t), \quad \varphi = C_2\exp(\mu\xi + \mathrm{i}\omega t)$$

that leads to the dispersion equation

$$D^{(A)}(\mu, \lambda_0, \varepsilon) = 4(1 + v)^2b_0^{-2}[b_0D^{(A)}(\mu, \lambda_0) + D_1^{(A)}(\mu, \lambda_0)\varepsilon^2] = 0, \tag{49}$$

where

$$D_0^{(A)}(\mu, \lambda_0) = D_0(\mu, \lambda_0), D_1^{(A)}(\mu, \lambda_0) = E_0G_0\{(1/3)E_0G_0\mu^6 - [(4/5)(1 + v)(G_0 - \lambda_0^2)$$

$$- \tfrac{1}{3}b_0\lambda_0^2]\mu^4 - \tfrac{4}{5}(1 + v)\lambda_0^2(G_0 - b_0\lambda_0^2)\mu^2\}.$$

From equation (49) the following groups of zeros of the function $D^{(A)}(\mu, \lambda_0, \varepsilon)$ can be reached

$$(1)\ \mu_k = \mu_{k0} + \mu_{k2}\varepsilon^2 + \ \cdots \quad (k = 1, 2), \quad D_0^{(A)}(\mu_{k0}, \lambda_0) = D_0(\mu_{k0}, \lambda_0) = 0, \tag{50}$$

$$\mu_{k2} = [6(\lambda_0^2 - G_0)b_0E_0G_0\mu_{k0}]^{-1}D_1^{(A)}(\mu_{k0}, \lambda_0) \neq [6(\lambda_0^2 - G_0)b_0E_0G_0\mu_{k0}]^{-1}D_1(\mu_{k0}, \lambda_0),$$

$$(2)\ \mu_k = \mu_{k0}\varepsilon^q + \mu_{k1}\varepsilon^{3q} + \ \cdots, \quad \lambda_0 = \Lambda\varepsilon^q, \quad q > 0, \quad \mu_{k0} = \ \pm \Lambda i(E_0G_0)^{-1/2}, \tag{51}$$

$$\mu_{k1} = \ \pm \mathrm{i}v_1^2\Lambda^3(1/2)(E_0G_0)^{-3/2},$$

$$(3)\ \ \mu_k = \varepsilon^{-1/2}(\mu_{k0} + \varepsilon\mu_{k2} + \cdots), \quad \mu_{k0}^4 + 3b_0(G_0 - \lambda_0^2)(E_0G_0)^{-1} = 0, \tag{52}$$

$$\mu_{k2} = v_2^2\lambda_0^2[4(G_0 - \lambda_0^2)\mu_{k0}]^{-1} + 3(1+v)(G_0 - \lambda_0^2)[5E_0G_0\mu_{k0}]^{-1}.$$

Comparison of equations (45)–(47) and (50)–(52) with corresponding expansions (13), (25), (18) from the three-dimensional theory show that the main terms of the expansions coincide, while the subsequent terms differ essentially. Note that in the case of the super-low-frequency vibrations the two first terms of the expansions coincide.

Numerical simulations were performed for hollow magnesium and cadmium cylinders with $v = 0{\cdot}357$, $v_1 = 0{\cdot}252$, $v_2 = 0{\cdot}226$, $G_0 = 1{\cdot}021$ and $v = 0{\cdot}116$, $v_1 = 0{\cdot}254$, $v_2 = 0{\cdot}722$, $G_0 = 2{\cdot}231$ respectively [17]. The aim of the analysis was to compare the roots μ of the dispersion equations (12) (the three-dimensional theory), (44) (the Kirchhoff– Love theory) and (49) (the Ambartsumjan theory) in order to estimate the relative accuracy of these applied theories. The range of the reduced frequency parameter $\lambda_0(\lambda^2 = 2(1+v)\lambda_0^2)$ variation was chosen taking for the following restrictions into account: $\lambda_0 \sim O(1)$, $\lambda_0^2 > G_0$, $\lambda_0^2 > G_0 b^{-1}$. This range corresponds to the case when the roots $\mu_k (k = 1, 2)$ of equations (12), (44), (49) are purely imaginary and have equal modulus, and between roots $\mu_k (k = 3–6)$ with equal modulus are two purely imaginary and two real roots (note that purely imaginary roots correspond to penetrating solutions of the equations of motion). The fact that for transversally isotropic material the relation $G = E/2(1+v)$ is valid reference [15] was taken into account.

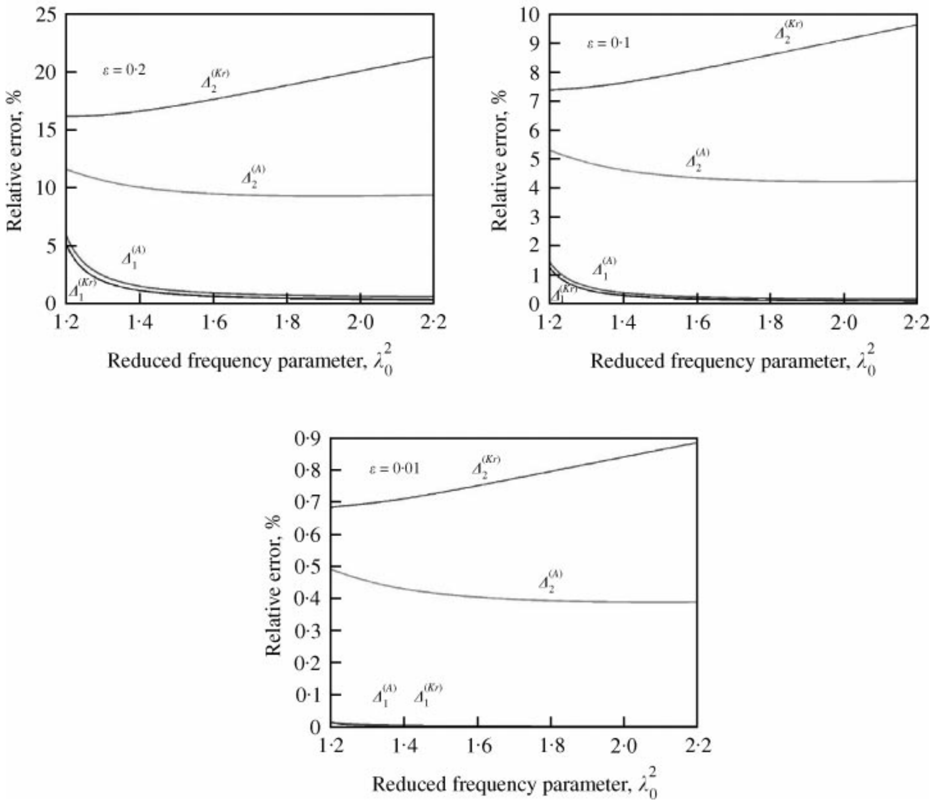

Figure 1. Precision of the applied Kirchhoff–Love and Ambartsumjan theories. Error estimations for the magnesium tube.

To characterize the relative accuracy of the applied theories the following parameters were introduced:

$$\Delta_1^{(Kr)} = \left|\frac{\mu_k - \mu_k^{(Kr)}}{\mu_k}\right| 100\% \quad (k = 1, 2); \quad \Delta_2^{(Kr)} = \left|\frac{\mu_k - \mu_k^{(Kr)}}{\mu_k}\right| 100\% \quad (k = 3\text{–}6);$$

$$\Delta_1^{(A)} = \left|\frac{\mu_k - \mu_k^{(A)}}{\mu_k}\right| 100\% \quad (k = 1, 2); \quad \Delta_2^{(A)} = \left|\frac{\mu_k - \mu_k^{(A)}}{\mu_k}\right| 100\% \quad (k = 3\text{–}6).$$

Here $\mu_k(k = 1, 2)$ and $\mu_k(k = 3\text{–}6)$ are the roots of equation (12), described by equations (13), (14) and (18), (19) respectively; $\mu_k^{(Kr)}(k = 1, 2)$ and $\mu_k^{(Kr)}(k = 3\text{–}6)$ are the roots of equation (44), described by equations (45) and (47) respectively; $\mu_k^{(A)}(k = 1, 2)$ and $\mu_k^{(A)}(k = 3\text{–}6)$ are the roots of equation (49), described by equations (50) and (52) respectively. Hence, the quantities $\Delta_1^{(Kr)}$, $\Delta_2^{(Kr)}$ and $\Delta_1^{(A)}$, $\Delta_2^{(A)}$ characterize the relative error of calculation of the roots of equation (12) on the basis of the Kirchhoff–Love and Ambartsumjan theories, correspondingly.

Figure 1 are presents results of calculations for the magnesium tube with different values of the non-dimensional wall-thickness parameter ε. Similar data for cadmium tube are presented in Figure 2.

It follows from Figure 1 that precision in determination of the first two roots (13) of equation (12) for magnesium tube on the basis of the Kirchhoff–Love and Ambartsumjan theories is practically identical while calculation of the four roots (18) of equation (12) on the

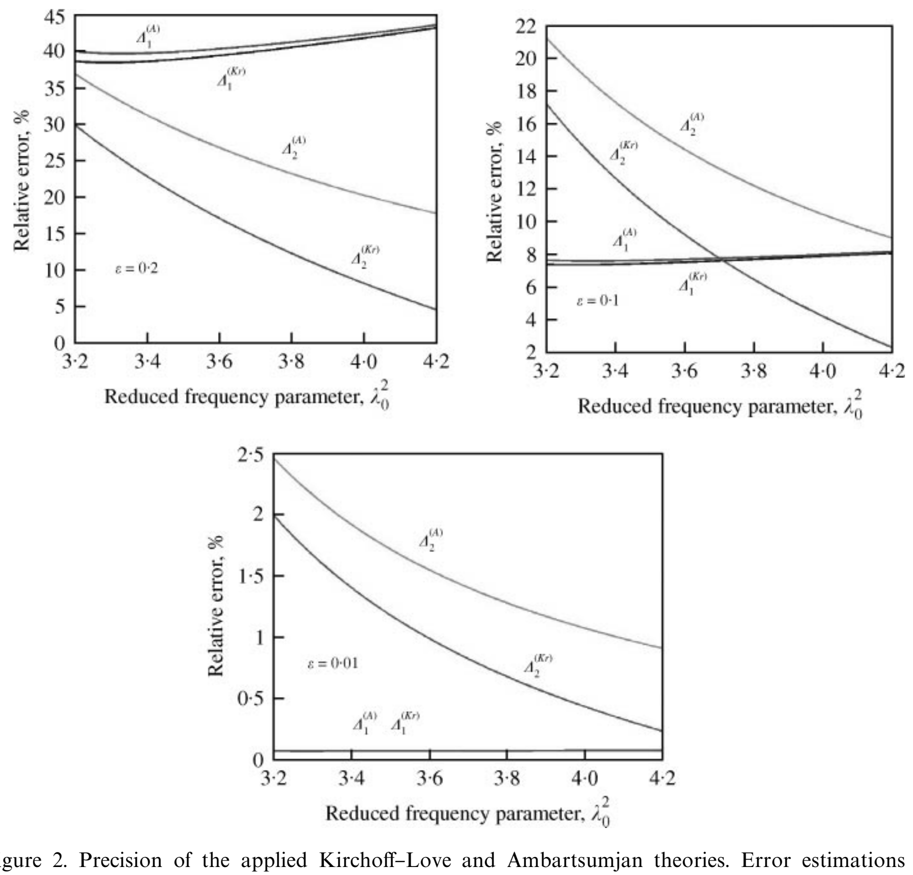

Figure 2. Precision of the applied Kirchoff–Love and Ambartsumjan theories. Error estimations for the cadmium tube.

basis the Ambartsumjan theory gives a smaller error than that on the basis of the Kirchhoff–Love theory. For the cadmium tubes (Figure 2) the same conclusion can be drawn with respect to the first two roots (13) of equation (12) while the data for the four roots (18) of equation (12) show that the Kirchhoff–Love theory is more exact. Figures 1 and 2 also show that for both materials the accuracy in determination of all six roots (13), (18) of equation (12) on the basis of the theories mentioned above increases with decreasing non-dimensional wall-thickness parameter ε.

The roots of the dispersion equation (12), defined by equations (21)–(23) and (36), cannot be found within the Kirchhoff–Love and Ambartsumjan theories and they demonstrate the qualitative difference between the theory of anisotropic and isotropic elastic shells. It is necessary to stress that both the results of reference [14] and the above analysis indicate that introducing of the correcting terms into equations of the classical two-dimensional shell theory cannot help to describe certain important dynamic phenomena for thin shells. Only an asymptotic analysis of the three-dimensional equations is able to describe them.

Note that in the partial case $G_0 = 1$ these results coincide with the results [12, 13], where the corresponding isotropic case has been considered.

6. CONCLUSION

The solution of the dynamic problem for a transverse isotropic hollow cylinder, loaded by axisymmetric harmonic loads at the butt ends and unloaded on the lateral surface, is given on the basis of the dynamic theory of elasticity. The resulting dispersion equation was studied by the asymptotic method and suggests that the cylinder is thin walled and the wall thickness, related to the radius of the shell middle surface, represents the small parameter of the problem.

The properties of the roots of the dispersion equation, characterizing the oscillation patterns in different frequency regions, were investigated. On the basis of this analysis the asymptotic representations of the equations integrals have been reached.

The asymptotic analysis made it possible to receive both solutions, corresponding to applied (two-dimensional) theories of shells, and others, which cannot be obtained on the basis of the latter. These solutions play an important role in describing of the stress–strain state in dynamics of the thin-walled cylinder.

The cases where the stress–strain states for isotropic and transverse isotropic thin-walled cylinders differ essentially, were established. It was shown that for certain relations between elastic constants of the material and the frequency parameter, the properties of the integrals of the equations of the dynamic theory of elasticity are changed.

The roots of the dispersion equation, within the three-dimensional theory, and others taken from applied Kirchhoff–Love and Ambartsumjan theories, are compared both analytically and numerically. It is shown that accuracy in approximate evaluation of the roots of the exact dispersion equation within the Kirchhoff–Love and Ambartsumjan theories, respectively, depends on the tube material.

Note that from the present study, there exists a certain group of roots of the three-dimensional dispersion equation, which is governed by the essential anisotropy of the material and cannot be included within both the classical Kirchhoff–Love and the improved Ambartsumjan theories.

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APPENDIX A: NOMENCLATURE

u_r, u_z	displacements of the cylinder in the transverse and longitudinal directions
u_ρ, u_ξ	non-dimensional displacements of the cylinder in the transverse and longitudinal directions
r_1, r_2	inner and outer radii of the hollow cylinder

r_0	radius of the tube middle surface, $r_0 = (r_1 + r_2)/2$
ξ, ρ	non-dimensional axial and radial co-ordinates
ρ_1, ρ_2	non-dimensional inner and outer radii of the hollow cylinder
ω	angular frequency
g	density of material
G, G_1, v, v_1, E, E_1	elastic constants
λ^2	frequency parameter, $\lambda^2 = gr_0^2\omega^2 G_1^{-1}$
λ_0^2	reduced frequency parameter, $\lambda_0^2 = \lambda^2/2(1+v)$
σ_r, σ_z, τ_{rz}	radial, longitudinal and shear components of the stress tensor
μ	the spectrum parameter
$J_n(x)$, $Y_n(x)$	Bessel's functions of the first and second kind and order n
ε	non-dimensional wall-thickness parameter, $\varepsilon = (\rho_1 - \rho_2)/2$
t	time
l	half of the cylinder's length
u_0, v_0	displacements of the shell middle surface in the longitudinal and transverse directions
h	thickness of the shell
$\varphi(\xi, t)$	the shear deformation function